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# $\mathcal{PT}$-symmetric potentials and the $so(2,2)$ algebra

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**Abstract**

Starting from a differential realization of the generators of the $so(2,2)$ algebra we connect the eigenvalue equation of the Casimir invariant either with the hypergeometric equation, or the Schrödinger equation. In the latter case we consider problems for which $so(2,2)$ appears as a potential algebra, connecting states with the same energy in different potentials. We analyse the role of the two $so(2,1)$ subalgebras and point out their importance for $\mathcal{PT}$-symmetric problems, where the doubling of bound states is known to occur. We present two mechanisms for this and illustrate them with the example of the Scarf and the Pöschl–Teller II potentials. We also analyse scattering states, transmission and reflection coefficients for these potentials.

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## 1. Introduction

Exactly solvable quantum mechanical potentials are often discussed using various linear differential operators which connect different solutions of the wave equations. A particularly interesting case is when the Hamiltonian can also be constructed from these operators. The underlying ideas behind these approaches might have different origins, but the applied technical methods are often rather similar.

One of the approaches of this kind is the factorization method [1], in which case the Schrödinger equation, which is a second-order differential equation, is factorized into two first-order operators, which are also Hermitian adjoints of each other. It follows from the construction that the first-order operators connect eigenstates of different Hamiltonians corresponding to the same energy eigenvalue. A reformulation of the factorization method was introduced some twenty years ago in the context of supersymmetric quantum mechanics (SUSYQM) [2]. The key element of this method is also the use of first-order differential operators connecting isoenergetic levels of different potentials.

A different method is the use of algebraic (group theoretical) techniques in the description of quantum mechanical potentials. In this case the ladder and weight operators typically appear as the elements of various algebras, while the Hamiltonian is related to the Casimir operator of the same algebra (although in some approaches its role is played by an element of the algebra). An important aspect of the algebraic approach is that the powerful mathematical methods of group theory can directly be applied to physical systems, and this often simplifies the discussion of these problems. The solutions of the wave equations are assigned to group representations. Generally bound-state solutions are considered. In some cases the states assigned to the same irreducible representation (irrep) are bound levels belonging to the same potential problem either with different energies or degenerate in energy. In these cases we talk about spectrum generating algebras and degeneracy algebras, respectively.

When all the states of a problem can be assigned to a single irrep of some algebra, it takes the name of dynamical algebra. Examples of this are $so(4,2)$ for the Coulomb potential [3] and $mp(6)$ for the isotropic three-dimensional harmonic oscillator [4]: the latter is the algebra of the metaplectic group, $Mp(6)$, which is the covering group of the symplectic group $Sp(6,\mathrm{R})$, containing the states with even numbers of oscillator quanta in one irrep and the states with odd numbers of quanta in another irrep.

A more recent development was the introduction of the concept of the potential algebra [5]. This is somewhat similar to the degeneracy algebra in the sense that the elements of the algebra connect degenerate levels which, however, belong to different Hamiltonians. Not surprisingly, the problems discussed in terms of the potential algebra context are essentially the same ones that can be approached using the factorization method and SUSYQM. The number of exactly solvable problems admitting a potential algebra is limited to some well-known (shape-invariant [6]) problems such as the Pöschl–Teller and Morse potentials, for example. The ladder operators of the potential algebras can be recognized as the shift operators of type A problems in the factorization method [1, 7] and also as the operators $A$ and $A^{\dagger}$ related to these problems in SUSYQM. The practical equivalence of the potential algebra and the SUSYQM approach to shape-invariant potentials has been discussed in [8], and later on in [9, 10].

A further interesting aspect of potential algebras is that whenever they are non-compact, scattering states can be treated on an equal footing with bound states: the former belong to the continuous unitary irreps (unirreps) and the latter to the discrete unirreps of the relevant algebras. Non-compact potential algebras are $so(2,1) \sim su(1,1)$ assigned to the Morse and the Pöschl–Teller potentials [5], but later on the larger $so(2,2)$ potential algebra was also introduced [11, 12]. In fact, $so(2,2)$ has been identified as the algebra of the rather general Natanzon family [13] of solvable potentials [11, 12].

Here we discuss a rather general differential realization of the $so(2,2)$ algebra and attempt to describe second-order differential equations by identifying them with the equation of the Casimir operator(s) of this algebra. Besides obtaining the Schrödinger equation, we also study the hypergeometric differential equation with this method. With this we generalize an earlier work of ours in which we considered a similar differential realization of the $su(1,1) \sim so(2,1)$ algebra to discuss shape-invariant potentials [8]. Our motivation is a recent result concerning the algebraic approach [14] to some so-called $\mathcal{PT}$ invariant potentials [15], which are complex, nevertheless typically possess real energy eigenvalues. In particular, we found that the bound (normalizable) states of the $\mathcal{PT}$ invariant version of the Scarf potential are doubled, in the sense that states that appear as resonances in the Hermitian version also become bound states with real energy eigenvalues [14]. This means that the $so(2,1)$ algebra associated with the usual bound states is also doubled, therefore the bound states of this problem now belong to a larger algebra which includes both $so(2,1)$ algebras. The doubling of bound states in $\mathcal{PT}$-symmetric potentials can occur via another mechanism too, i.e. by regularizing their singularities [16]. This naturally raises the question whether this process also has algebraic implications.

In section 2 we present the differential realization of the $so(2,2)$ algebra using a formalism which allows the $so(4)$ and, in principle, the $so(3) \oplus so(2,1)$ algebras too, and derive the Schrödinger equation as well as the hypergeometric differential equation. In section 3 we discuss the role of the ladder operators and their relation to the doubling of normalizable states, while in section 4 we present our conclusions.

## 2. A differential realization of some six-parameter Lie algebras

Here we give a constructive differential realization of the six-parameter algebras $so(2,2)$, $so(4)$ and $so(3) \oplus so(2,1)$. We follow the presentation of a similar paper [8], where a realization of the $su(1,1)$ algebra was introduced in terms of first-order linear differential operators. In this construction we start with generators containing altogether ten functions, which reduce to two after enforcing the fulfilment of the appropriate commutation relations.

Let us consider the construction of two sets of operators

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm h_1(x)\frac{\partial}{\partial x} \pm g_1(x) + f_1(x)J_z + c_1(x) + k_1(x)K_z\right) \tag{1}$$

$$J_z = -\mathrm{i}\frac{\partial}{\partial\phi} \tag{2}$$

and

$$K_\pm = \mathrm{e}^{\pm\mathrm{i}\chi}\left(\pm h_2(x)\frac{\partial}{\partial x} \pm g_2(x) + f_2(x)J_z + c_2(x) + k_2(x)K_z\right) \tag{3}$$

$$K_z = -\mathrm{i}\frac{\partial}{\partial\chi}. \tag{4}$$

Now impose the commutation relations

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2aJ_z \tag{5}$$

$$[K_z, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2bK_z \tag{6}$$

$$[J_i, K_j] = 0 \qquad i, j = +, -, z. \tag{7}$$

For $a = b = 1$ we get the $so(2,1) \oplus so(2,1)$ algebra, while for $a = 1$, $b = -1$ and $a = b = -1$ we get the $so(3) \oplus so(2,1)$ and $so(3) \oplus so(3)$ algebras, respectively. It is well known [3] that the first algebra is isomorphic to $so(2,2)$ and the third algebra to $so(4)$. In particular, Hermitian potentials connected with $so(2,2)$ as a potential algebra have been studied in [11, 12]. Our approach is more general, in the sense that it permits treatment of other six-parameter algebras, such as $so(4)$ and $so(3) \oplus so(2,1)$, and is extended to non-Hermitian $\mathcal{PT}$-symmetric potentials. The above relations lead to a system of first-order differential equations for the functions appearing in equations (1) and (3), which reduces to the following relations after some tedious but straightforward calculations:

$$k_2^2 - h_2 k_2' = b \qquad h_2 f_2' - f_2 k_2 = 0 \qquad k_2^2 - f_2^2 = b \tag{8}$$

$$c_1 = c_2 = 0 \tag{9}$$

$$h_1 = A h_2 \qquad f_1 = A k_2 \qquad k_1 = A f_2 \qquad g_1 = A g_2 \tag{10}$$

$$A^2 = \frac{a}{b} = \pm 1. \tag{11}$$

Here we have assumed that $h_i(x) \neq 0$, $k_i \neq 0$ and $f_i \neq 0$ holds. For $h_i(x) = 0$ the differential term with respect to $x$ would be cancelled in $J_\pm$ (1) and $K_\pm$ (3), while $f_i(x) = k_i(x) = 0$ would contradict equation (8). We also note that from the three equations in (8) only two are independent and that the choice of $h_2(x)$ determines $f_2(x)$ and $k_2(x)$ immediately. However $h_2(x)$ does not determine $g_2(x)$, so there are two independent functions defining this construction.

The Casimir invariant

$$C_2^{(JK)} = 2C_2^{(J)} + 2C_2^{(K)} \equiv 2\left(-aJ_+J_- + J_z^2 - J_z - bK_+K_- + K_z^2 - K_z\right) \tag{12}$$

is a second-order differential operator

$$C_2^{(JK)}\Psi = 4bh_2^2\Psi'' + 4bh_2(h_2' + 2g_2 - k_2)\Psi' + \left[4b\left(h_2 g_2' + g_2^2 - k_2 g_2\right) + 2\left(1 - bk_2^2 - bf_2^2\right)\left(J_z^2 + K_z^2\right) - 8bf_2k_2J_zK_z\right]\Psi. \tag{13}$$

The eigenfunctions of $C_2^{(JK)}$, which are also the eigenfunctions of $J_z$ and $K_z$, are $\Psi \equiv \Psi(x, \phi, \chi) = \mathrm{e}^{\mathrm{i}(m\phi + m'\chi)}\psi(x)$. Here $\psi(x)$ is the physical wavefunction depending on the coordinate $x$, while $\phi$ and $\chi$ are auxiliary variables, which are multiplied with $m$ and $m'$, the eigenvalues of generators $J_z$ and $K_z$, respectively.

Since the above algebras are of rank 2, they admit a second Casimir invariant, which can be written as the difference of the two $SO(2,1)$ Casimir invariants in (12)

$$\tilde{C}_2^{(JK)} = 2C_2^{(J)} - 2C_2^{(K)}. \tag{14}$$

It turns out that the eigenvalue of this operator is always zero for the present differential realization of the algebra, irrespective of $a$ and $b$. Therefore, we have generated the symmetric irrep of $so(2,2)$ (or $so(4)$) [12], usually labelled as $(\omega, 0)$, where $\omega$ is the quantum number defining the eigenvalue of the first Casimir invariant

$$C_2^{(JK)}\Psi = \omega(\omega + 2)\Psi. \tag{15}$$

$\omega$ is connected with the eigenvalue $j(j+1)$ of the Casimir invariant of $so(2,1)$ (or $so(3)$) by the relation $\omega = 2j$. Of course, a simple formal transition from an $so(2,1)$ algebra to an $so(3)$ algebra can be made by multiplying the $h_i, g_i, f_i, k_i, c_i$ $(i = 1, 2)$ functions with the imaginary unit i. This exactly corresponds to the changes $a \to -1$ and $b \to -1$. This simultaneous change of the two subalgebras corresponds to $A = \pm 1$. In principle, we can try to change only one of the $so(2,1)$ subalgebras into $so(3)$ (i.e. to take $A = \pm\,\mathrm{i}$), but it turns out that this complicates the structure of the differential equation obtained from (15) so much that it no longer becomes solvable in general.

We note that the $f_2 = 0$ choice is also possible in the set of equations (8), but then the other functions will give less rich structure to the generators. In that case only the $h_i$ and $g_i$ will be non-constant functions:

$$\begin{aligned} h_1 &= Ah_2 \qquad g_1 = Ag_2 + D \qquad k_2 = \pm b^{1/2} \\ f_1 &= \pm a^{1/2} \qquad k_1 = f_2 = c_1 = c_2 = 0. \end{aligned} \tag{16}$$

The generators (1) and (3) transform in a characteristic way with respect to variable transformations $x \to z$

$$\begin{aligned}
h_i(x) &\to h_i^{\mathrm{v}}(z) = h_i(x(z))\frac{\mathrm{d}z}{\mathrm{d}x} \\
g_i(x) &\to g_i^{\mathrm{v}}(z) = g_i(x(z)) \\
f_i(x) &\to f_i^{\mathrm{v}}(z) = f_i(x(z)) \\
k_i(x) &\to k_i^{\mathrm{v}}(z) = k_i(x(z)) \\
c_i(x) &\to c_i^{\mathrm{v}}(z) = c_i(x(z))
\end{aligned} \tag{17}$$

and similarity transformations

$$J_\alpha \to J_\alpha^{\mathrm{s}} = \mathcal{F} J_\alpha \mathcal{F}^{-1} \tag{18}$$

governed by $\mathcal{F} = 1/v(x)$:

$$\begin{aligned}
h_i(x) &\to h_i^{\mathrm{s}}(x) = h_i(x) \\
g_i(x) &\to g_i^{\mathrm{s}}(x) = g_i(x) + h_i(x)\frac{\mathrm{d}}{\mathrm{d}x}\ln v(x) \\
f_i(x) &\to f_i^{\mathrm{s}}(x) = f_i(x) \\
k_i(x) &\to k_i^{\mathrm{s}}(x) = k_i(x) \\
c_i(x) &\to c_i^{\mathrm{s}}(x) = c_i(x).
\end{aligned} \tag{19}$$

These results are similar to those in [8] derived for $su(1,1) \sim so(2,1)$ algebras related to the same problem. These generators are obtained from $J_+$ and $J_-$ in equation (1) by replacing $c_1$ with $k_1$ and then dropping the $K_z$ term. We are going to discuss the importance of the two distinct algebras later on in this section.

### 2.1. The Schrödinger equation

If we want to cancel the linear derivative term in (13), then we can make use of the freedom of choosing $g_2(x)$ and prescribe

$$g_2 = \tfrac{1}{2}(k_2 - h_2') \tag{20}$$

using a similarity transformation (19). With this choice we get

$$\begin{aligned}
C_2^{(JK)}\Psi &= 4bh_2^2\Psi'' + \left[b\left((h_2')^2 + k_2^2 - 2h_2''h_2\right) - 2 + 4\left(1 - bk_2^2\right)\left(J_z^2 + K_z^2\right) - 8bf_2k_2J_zK_z\right]\Psi \\
&= \omega(\omega+2)\Psi.
\end{aligned} \tag{21}$$

A Schrödinger-type differential equation can be obtained from equation (21) for constant $h_2$ as

$$-\frac{1}{4bh_2^2}\left[C_2^{(JK)} - \omega(\omega+2)\right]\Psi = 0 \tag{22}$$

after substituting the eigenvalues $m$ and $m'$ of the operators $J_z$ and $K_z$. Equation (22) implies that in this case the energy eigenvalues are related to the eigenvalues of the Casimir operator. In fact, the energy eigenvalues are $E = -(\omega+1)^2/\left(4bh_2^2\right)$, and the Hamiltonian is related to $C_2^{(JK)}$ via $H = -\left(C_2^{(JK)} + 1\right)/\left(4bh_2^2\right)$.

Some exactly solvable shape-invariant potentials can be derived with the algebraic construction described previously. These potentials all belong to the PI class in the classification used in [8, 17], and the algebra corresponding to them is either $so(2,2)$ ($a = b = 1$) or $so(4)$ ($a = b = -1$). In this subsection we focus on the members of the PI potential class. There is some ambiguity in the names of these potentials; here we follow the notation of figure 5.1 in [2].

The hyperbolic type Scarf (or Gendenshtein) potential (class PI(i sinh($x$)) [17]) with $so(2,2)$ generators is obtained by substituting $h_2 = 1$, $g_2 = -\frac{1}{2}\tanh x$, $f_2 = \mathrm{i}/\cosh x$, $k_2 = -\tanh x$ and $a = b = 1$:

$$V(x) = -\left(m^2 + m'^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 x} - 2\mathrm{i}mm'\frac{\sinh x}{\cosh^2 x} \tag{23}$$

and the $so(2,2)$ generators are

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - \tanh x\left(J_z \pm \frac{1}{2}\right) + \frac{\mathrm{i}}{\cosh x}K_z\right) \tag{24}$$

$$K_\pm = \mathrm{e}^{\pm\mathrm{i}\chi}\left(\pm\frac{\partial}{\partial x} - \tanh x\left(K_z \pm \frac{1}{2}\right) + \frac{\mathrm{i}}{\cosh x}J_z\right). \tag{25}$$

We are going to study this potential in detail in the next section. It is to be stressed that in the case $m$ and $m'$ are real, potential (23) is not Hermitian, but $\mathcal{PT}$-symmetric, i.e. invariant under simultaneous parity reflection and time reversal, and satisfies the condition $V(x) = V^*(-x)$.

Generalized Pöschl–Teller potential (class PI(cosh($x$)) [17, 26, 27] with $so(2,2)$ generators:

$$V(x) = \left(m^2 + m'^2 - \frac{1}{4}\right)\frac{1}{\sinh^2 x} - 2mm'\frac{\cosh x}{\sinh^2 x} \tag{26}$$

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - \coth x\left(J_z \pm \frac{1}{2}\right) + \frac{1}{\sinh x}K_z\right) \tag{27}$$

$$K_\pm = \mathrm{e}^{\pm\mathrm{i}\chi}\left(\pm\frac{\partial}{\partial x} - \coth x\left(K_z \pm \frac{1}{2}\right) + \frac{1}{\sinh x}J_z\right). \tag{28}$$

The Pöschl–Teller II potential (class PI(cosh($2x$)) [17]) with $so(2,2)$ generators:

$$V(x) = -\left((m - m')^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 x} + \left((m + m')^2 - \frac{1}{4}\right)\frac{1}{\sinh^2 x} \tag{29}$$

$$J_\pm = \frac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - (\coth x + \tanh x)\left(J_z \pm \frac{1}{2}\right) + (\tanh x - \coth x)K_z\right) \tag{30}$$

$$K_\pm = \frac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}\chi}\left(\pm\frac{\partial}{\partial x} - (\coth x + \tanh x)\left(K_z \pm \frac{1}{2}\right) + (\tanh x - \coth x)J_z\right). \tag{31}$$

This potential is basically equivalent to the generalized Pöschl–Teller potential (26) (it can be obtained from it by a variable transformation $x \to x/2$) and has been discussed in [11, 18, 19], for example, together with the corresponding $so(2,2)$ algebra, as a problem on the real half-line, owing to the singularity at the origin. We shall study a regularized $\mathcal{PT}$-symmetric version of it in the next section.

Trigonometric type Scarf potential (class PI(cos($x$)) [17]) with $so(4)$ generators:

$$V(x) = \left(m^2 + m'^2 - \frac{1}{4}\right)\frac{1}{\sin^2 x} - 2mm'\frac{\cos x}{\sin^2 x} \tag{32}$$

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - \cot x\left(J_z \pm \frac{1}{2}\right) + \frac{1}{\sin x}K_z\right) \tag{33}$$

$$K_{\pm} = e^{\pm i\chi}\left(\pm\frac{\partial}{\partial x} - \cot x\left(K_z \pm \frac{1}{2}\right) + \frac{1}{\sin x}J_z\right). \tag{34}$$

Pöschl–Teller I potential (class PI($\cos(2x)$) [17]) with $so(4)$ generators:

$$V(x) = \left((m+m')^2 - \frac{1}{4}\right)\frac{1}{\sin^2 x} + \left((m-m')^2 - \frac{1}{4}\right)\frac{1}{\cos^2 x} \tag{35}$$

$$J_{\pm} = \frac{1}{2}e^{\pm i\phi}\left(\pm\frac{\partial}{\partial x} + (\tan x - \cot x)\left(J_z \pm \frac{1}{2}\right) - (\cot x + \tan x)K_z\right) \tag{36}$$

$$K_{\pm} = \frac{1}{2}e^{\pm i\chi}\left(\pm\frac{\partial}{\partial x} + (\tan x - \cot x)\left(K_z \pm \frac{1}{2}\right) - (\cot x + \tan x)J_z\right). \tag{37}$$

Again, this is an equivalent form of equation (32). This potential and the relevant $so(4)$ algebra have been discussed in [20], and later on in [21], where the $so(4)$ potential algebra has been extended to an $sl$(4,R) dynamical potential algebra. Some elements of this latter algebra are beyond the differential realization we considered here.

The potentials (23), (26), (29), (32) and (35) have the common feature that the algebras assigned to them are all potential algebras. As discussed in [8], this is related to the fact that the $h_i(x)$ functions appearing in the generators (1) (and (3)) are constants, and consequently the coefficient of the second-order derivative term in the equation of the Casimir operator (13) is also constant, so this equation directly becomes Schrödinger-like. Obviously, the generators $J_{\pm}$ and $K_{\pm}$ then ladder between states with the same energy, so they are elements of a potential algebra and can easily be related to SUSYQM generators which act similarly [8]. In the language of the factorization method [1, 7] this corresponds to type A factorization, as discussed in [17], for example. When the $h_i(x)$ functions are not constants, then equation (13) is not Schrödinger-like and the generators ladder between levels belonging to different energies. In some of these cases one naturally gets spectrum generating algebras from this systematic procedure [8].

Finally we mention that it is also possible to derive the Schrödinger equation for the $f_2 = 0$ case, as in equation (16). Assuming again $g_1 = (f_1 - h'_1)/2$, $g_2 = (k_2 - h'_2)/2$ and $D = 0$, the equation of the Casimir operator is

$$\begin{aligned} C_2^{(JK)}\Psi &= 4bh_2^2\Psi'' + \left[4b\left(-\tfrac{1}{2}h_2h''_2 + \tfrac{1}{4}(h'_2)^2\right) - 1\right]\Psi \\ &= \omega(\omega+2)\psi. \end{aligned} \tag{38}$$

However, it turns out that although it is possible to obtain Schrödinger-like equations from (38), this more restricted construction is not flexible enough to accommodate all the solutions in the same algebraic framework.

### *2.2. The hypergeometric differential equation*

We can define the $h_i, g_i, f_i$ and $k_i$ functions such that equation (13) reproduces the hypergeometric equation

$$z(1-z)\frac{d^2F}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{dF}{dz} - \alpha\beta F = 0 \tag{39}$$

rather than the Schrödinger equation. If this turns out to be possible, then obviously, the Schrödinger equation with solvable potentials related to the hypergeometric equation (i.e. Natanzon potentials) can also be related to this algebra by means of variable and similarity transformations (17) and (19).

Equations (8) indicate that choosing $h_2(z)$ determines all the other unknown functions. A natural choice for $h_2(z)$ is

$$h_2(z) = dz^p(1-z)^q. \tag{40}$$

Starting with $d = b^{-1/2}$, $p = 1/2$ and $q = 1$, i.e. $h_2 = b^{-1/2}z^{1/2}(1-z)$ and setting $b = 1$ we get

$$\begin{aligned} h_2(z) &= Ah_1(z) = z^{1/2}(1-z) \\ k_2(z) &= Af_1(z) = -\tfrac{1}{2}z^{-1/2}(1+z) \\ f_2(z) &= Ak_1(z) = \tfrac{1}{2}z^{-1/2}(1-z). \end{aligned} \tag{41}$$

To get the hypergeometric equation, one has to set $g_i(z)$ as

$$g_2(z) = Ag_1(z) = \frac{\gamma-1}{2}z^{-1/2} - \frac{\alpha+\beta}{2}z^{1/2}. \tag{42}$$

With this choice the equation of the $C_2^{(J,K)}$ Casimir operator (15) becomes $4(1-z)$ times equation (39) with the basis functions

$$\Psi = \mathrm{e}^{\mathrm{i}(m\phi+m'\chi)}F(\alpha,\beta;\gamma;z) \tag{43}$$

provided that the parameters $\alpha$, $\beta$ and $\gamma$ are related to the group representation labels as

$$(\gamma-1)^2 = (m-m')^2 \qquad (\alpha-\beta)^2 = (m+m')^2 \qquad (\alpha+\beta-\gamma)^2 = (\omega+1)^2. \tag{44}$$

This is similar to equation (3.1.21) in [22], although in that realization of the algebra $m_1$ and $m_2$ correspond to our $m+m'$ and $m-m'$.

For $b = -1$, trigonometric functions appear in the $f_i(z)$, $k_i(z)$ and $g_i(z)$ functions and they cannot be cancelled from (13), so one cannot get the hypergeometric equation from it.

The ladder operators are

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}z^{-1/2}\left(\pm z(1-z)\frac{\partial}{\partial z} \pm \frac{\gamma-1}{2} \mp \frac{\alpha+\beta}{2}z - \frac{1}{2}(1+z)J_z + \frac{1}{2}(1-z)K_z\right) \tag{45}$$

$$K_\pm = \mathrm{e}^{\pm\mathrm{i}\chi}z^{-1/2}\left(\pm z(1-z)\frac{\partial}{\partial z} \pm \frac{\gamma-1}{2} \mp \frac{\alpha+\beta}{2}z + \frac{1}{2}(1-z)J_z - \frac{1}{2}(1+z)K_z\right). \tag{46}$$

This realization is different from those in [7] for several reasons. First, there is only one set of generators related to a $G(1,0)$ algebra, which corresponds to our $so(2,1)$ or $so(3)$. Second, the equivalents of our ladder operators $J_+$ and $J_-$ would contain different $h_1$ functions.

Now we can assume that

$$(\gamma-1) = m-m' \qquad \alpha-\beta = m+m' \tag{47}$$

i.e.

$$m = \tfrac{1}{2}(\alpha-\beta+\gamma-1) \qquad m' = \tfrac{1}{2}(\alpha-\beta-\gamma+1). \tag{48}$$

The other choices of the signs due to taking the square roots in equation (44) simply correspond to exchanging $J_\pm$ with $K_\pm$ and/or $\alpha$ with $\beta$.

One can show that the effect of the ladder operators is the following:

$$\begin{aligned} J_+\psi_{m,m'}(z) &= \frac{\alpha(\beta-\gamma)}{\gamma}z^{1/2}\psi_{m+1,m'}(z) \\ &= \frac{\alpha(\beta-\gamma)}{\gamma}z^{1/2}\,\mathrm{e}^{\mathrm{i}[(m+1)\phi+m'\chi]}F(\alpha+1,\beta;\gamma+1;z) \end{aligned} \tag{49}$$

$$\begin{aligned} J_-\psi_{m,m'}(z) &= (1-\gamma)z^{-1/2}\psi_{m-1,m'}(z) \\ &= (1-\gamma)z^{-1/2}\,\mathrm{e}^{\mathrm{i}[(m-1)\phi+m'\chi]}F(\alpha-1,\beta;\gamma-1;z) \end{aligned} \tag{50}$$

$$K_+\psi_{m,m'}(z) = (\gamma-1)z^{-1/2}\psi_{m,m'+1}(z) = (\gamma-1)z^{-1/2}\,e^{i[m\phi+(m'-1)\chi]}F(\alpha,\beta-1;\gamma-1;z) \tag{51}$$

$$K_-\psi_{m,m'}(z) = \frac{\beta(\gamma-\alpha)}{\gamma}z^{1/2}\psi_{m,m'-1}(z) = \frac{\beta(\gamma-\alpha)}{\gamma}z^{1/2}\,e^{i[m\phi+(m'-1)\chi]}F(\alpha,\beta+1;\gamma+1;z). \tag{52}$$

Equations (49)–(52) show that the ladder operators change the indices $m, m'$ and variables $\alpha, \beta, \gamma$ in the expected way, however, the right-hand sides of these equations contain a $z$-dependent factor in addition to the basis functions (43). Nevertheless, direct calculation shows that the basis functions (43) satisfy the eigenequation (15) of the Casimir operator.

Other choices of $d$, $p$ and $q$ could also be made in equation (40), and they might also lead to the hypergeometric differential equation with other realizations.

Finally, in [12, 22] it was shown that an $so(2,1)$ algebra related to the confluent hypergeometric equation can also be obtained from the $so(2,2)$ algebra by using a contraction mechanism. We also find that the differential realizations (1) and (3) are not suitable to derive the confluent hypergeometric equation, however, the $so(2,1)$ algebra related to it can be obtained from (1) with $h_1 = z$, $g_1 = (c-z)/2$, $f_1 = 1$, $k_1 = 0$, $c_1 = -z/2$ and $a = 1$. In this case the basis functions are $e^{im\phi}F(\alpha,\gamma;z)$ with $\alpha = -j+m$ and $\gamma = -2j$.

## 3. Algebraic treatment of $\mathcal{PT}$-symmetric potentials

Let us study now the role of the two subalgebras and in particular, that of the ladder operators $J_\pm$ and $K_\pm$. Their structure is rather similar, and we saw that they systematically change the $m$ and $m'$ representation labels in the two independent solutions. In normal circumstances (i.e. Hermitian problems), only one of the solutions is regular, which might raise the question whether we face some kind of redundancy. However, as we shall see below, the two solutions can *both* be defined to be regular when the potentials are not Hermitian anymore; nevertheless, they satisfy the requirement of $\mathcal{PT}$ symmetry [23]. For one-dimensional potentials this invariance means $V^*(-x) = V(x)$. With the example of the hyperbolic Scarf and the Pöschl–Teller II potentials we demonstrate below that there are two essentially different ways to obtain the doubling of regular (real-energy bound-state) solutions, applicable to different types of potentials. In the first case the potential is regular even in the Hermitian case, and the second set of regular solutions develops from resonance solutions when $\mathcal{PT}$ symmetry is prescribed. This scenario has already been studied using the smaller $su(1,1) \sim so(2,1)$ algebra and, consequently, only in the presence of a single set of ladder operators [14]. In the second case the Hermitian potential is singular at $x = 0$, however, it can be regularized and extended to the full $x$ axis by a formal imaginary coordinate shift [24], which breaks Hermiticity, while respecting $\mathcal{PT}$ symmetry.

This imaginary shift, $x \to x + i\epsilon$, destroys the $\mathcal{P}$ symmetry of the hyperbolic and trigonometric functions appearing in the examples of section 2.1, nevertheless, these functions maintain definite symmetry with respect to $\mathcal{PT}$ symmetry [24]. In fact, this also determines the way the generators (1), (2), (3) and (4) transform under the $\mathcal{PT}$ operation. Introducing a tilde to indicate that the generators now depend on the $\epsilon$ parameter, we find that

$$\mathcal{PT}(\tilde{J}/\tilde{K})_\pm(\mathcal{PT})^{-1} = (\tilde{J}/\tilde{K})_\mp \tag{53}$$

$$\mathcal{PT}(\tilde{J}/\tilde{K})_z(\mathcal{PT})^{-1} = -(\tilde{J}/\tilde{K})_z. \tag{54}$$

These relations hold whenever the $h_i(x)$ and $c_i(x)$ functions appearing in (1) and (3) are even under the $\mathcal{PT}$ operation, while the others, $g_i(x)$, $f_i(x)$ and $k_i(x)$, are odd. Also note that the $\mathcal{PT}$ operation has the same effect on the two ladder operators as Hermitian conjugation in the standard Hermitian case; this property is valid within the framework of the direct sum of the two $J$ and $K$ subalgebras, but not in the simpler $so(2,1)$ case of [14]. In that case the role of $K_z$ is played by a real number, and these quantities obviously transform in a different way under the $\mathcal{PT}$ operation.

### 3.1. The hyperbolic Scarf potential

Let us first analyse the (hyperbolic) Scarf potential, writing it in a more traditional form:

$$V(x) = (\lambda^2 - s(s+1))\frac{1}{\cosh^2 x} + \lambda(2s+1)\frac{\sinh x}{\cosh^2 x}. \tag{55}$$

Obviously, potential (23) is obtained from this by the substitutions $s = m - 1/2$ and $\lambda = -\mathrm{i}m'$. The two independent solutions are then

$$F_1(x) = (1+\mathrm{i}y)^{-\frac{s-\mathrm{i}\lambda}{2}}(1-\mathrm{i}y)^{-\frac{s+\mathrm{i}\lambda}{2}} F\left(-s-\mathrm{i}k, -s+\mathrm{i}k, \mathrm{i}\lambda - s + \frac{1}{2}; \frac{1+\mathrm{i}y}{2}\right) \tag{56}$$

and

$$F_2(x) = (1+\mathrm{i}y)^{\frac{s+1-\mathrm{i}\lambda}{2}}(1-\mathrm{i}y)^{-\frac{s+\mathrm{i}\lambda}{2}} F\left(\frac{1}{2} - \mathrm{i}\lambda - \mathrm{i}k, \frac{1}{2} - \mathrm{i}\lambda + \mathrm{i}k, s + \frac{3}{2} - \mathrm{i}\lambda; \frac{1+\mathrm{i}y}{2}\right) \tag{57}$$

with $y = \mathrm{i}\sinh x$. As discussed in [14], these solutions lead to discrete eigenvalues when $k = \mathrm{i}(s-n)$ and $k = \pm\lambda - \mathrm{i}\left(n + \frac{1}{2}\right)$ holds for the two solutions, respectively.

Multiplying (56) and (57) with the usual phase factors, we find that the effect of the ladder operators on the $\Psi_1(m, m'; x) = \mathrm{e}^{\mathrm{i}(m\phi + m'\chi)}F_1(x)$ and $\Psi_2(m, m'; x) = \mathrm{e}^{\mathrm{i}(m\phi + m'\chi)}F_2(x)$ basis functions is that $J_+$ ($J_-$) increases (decreases) $s$ and therefore $m$ by one unit, while $K_+$ ($K_-$) does the same with $\mathrm{i}\lambda$ and $m'$. As an illustration we present here the effect of $J_\pm$ and $K_\pm$ on the function $\Psi_1(m, m'; x)$:

$$\begin{aligned} J_+\Psi_1(m, m'; x) &= 2\mathrm{i}\left(\mathrm{i}\lambda - (s+1) + \frac{1}{2}\right)\mathrm{e}^{\mathrm{i}[(m+1)\phi + m'\chi]}(1+\mathrm{i}y)^{-\frac{(s+1)-\mathrm{i}\lambda}{2}}(1-\mathrm{i}y)^{-\frac{(s+1)+\mathrm{i}\lambda}{2}} \\ &\quad\times F\left(-(s+1) - \mathrm{i}k, -(s+1) + \mathrm{i}k, \mathrm{i}\lambda - (s+1) + \frac{1}{2}; \frac{1+\mathrm{i}y}{2}\right) \\ &= 2\mathrm{i}(m' - m)\Psi_1(m+1, m'; x) \end{aligned} \tag{58}$$

$$\begin{aligned} J_-\Psi_1(m, m'; x) &= -\frac{\mathrm{i}}{2}\frac{(-s-\mathrm{i}k)(-s+\mathrm{i}k)}{\mathrm{i}\lambda - s + \frac{1}{2}}\mathrm{e}^{\mathrm{i}[(m-1)\phi + m'\chi]}(1+\mathrm{i}y)^{-\frac{(s-1)-\mathrm{i}\lambda}{2}}(1-\mathrm{i}y)^{-\frac{(s-1)+\mathrm{i}\lambda}{2}} \\ &\quad\times F\left(-(s-1) - \mathrm{i}k, -(s-1) + \mathrm{i}k, \mathrm{i}\lambda - (s-1) + \frac{1}{2}; \frac{1+\mathrm{i}y}{2}\right) \\ &= -\frac{\mathrm{i}}{2}\frac{\left(-m + \frac{1}{2} - \mathrm{i}k\right)\left(-m + \frac{1}{2} + \mathrm{i}k\right)}{m' - m + 1}\Psi_1(m-1, m'; x) \end{aligned} \tag{59}$$

$$\begin{aligned} K_+\Psi_1(m, m'; x) &= \mathrm{i}\frac{\left(\frac{1}{2} + \mathrm{i}\lambda - \mathrm{i}k\right)\left(\frac{1}{2} + \mathrm{i}\lambda + \mathrm{i}k\right)}{-s + \frac{1}{2} + \mathrm{i}\lambda}\mathrm{e}^{\mathrm{i}[m\phi + (m'+1)\chi]}(1+\mathrm{i}y)^{-\frac{s-(\mathrm{i}\lambda+1)}{2}}(1-\mathrm{i}y)^{-\frac{s+(\mathrm{i}\lambda+1)}{2}} \\ &\quad\times F\left(-s - \mathrm{i}k, -s + \mathrm{i}k, -s + \frac{1}{2} + (\mathrm{i}\lambda + 1); \frac{1+\mathrm{i}y}{2}\right) \\ &= \mathrm{i}\frac{\left(m' + \frac{1}{2} - \mathrm{i}k\right)\left(m' + \frac{1}{2} - \mathrm{i}k\right)}{m' - m + 1}\Psi_1(m, m'+1; x). \end{aligned} \tag{60}$$

$$
\begin{aligned}
K_-\Psi_1(m,m';x) &= -\mathrm{i}\left(-s+\frac{1}{2}+(\mathrm{i}\lambda-1)\right)\mathrm{e}^{\mathrm{i}[m\phi+(m'-1)\chi]}(1+\mathrm{i}y)^{-\frac{s-(\mathrm{i}\lambda-1)}{2}}(1-\mathrm{i}y)^{-\frac{s+(\mathrm{i}\lambda-1)}{2}} \\
&\quad\times F\left(-s-\mathrm{i}k,-s+\mathrm{i}k,-s+\frac{1}{2}+(\mathrm{i}\lambda-1);\frac{1+\mathrm{i}y}{2}\right) \\
&= \mathrm{i}(m-m')\Psi_1(m,m'-1;x). \qquad (61)
\end{aligned}
$$

Similar relations hold for $\Psi_2(m,m';x)$ too. These can be obtained observing that the two independent solutions (56) and (57) are interrelated by the $s \leftrightarrow \mathrm{i}\lambda - \frac{1}{2}$, i.e. the $m \leftrightarrow m'$ replacement [14]. More precisely, the action of the $so(2,2)$ generators on $\Psi_2$ is obtained from equations (58) to (61) by the following replacements: $J \leftrightarrow K$, $m \leftrightarrow m'$, $\Psi_1(a,b;x) \rightarrow \Psi_2(b,a;x)$.

When the potential (55) is Hermitian (i.e. $\lambda$ is real), then only (56) leads to bound-state (normalizable) solutions, and the energy eigenvalues are $E_n = k^2 = -(s-n)^2$. In fact, in this case the second set of solutions corresponds to resonances [14], with energies $E_n = k^2 = \lambda^2 - \left(n+\frac{1}{2}\right)^2 \pm \mathrm{i}\lambda(2n+1)$. In [14] this was interpreted in the following way. The bound-state solutions form a basis for one of the $su(1,1) \sim so(2,1)$ algebras and belong to a discrete unitary representation of it, while resonance solutions are assigned to the non-unitary discrete irreps of the other $su(1,1) \sim so(2,1)$ algebra [25].

Remarkably, there is a situation when *both* (56) and (57) lead to normalizable solutions. For this, $\lambda$ has to be chosen imaginary such that $n + \frac{1}{2} < \mathrm{i}\lambda$ holds. This means that the second set of energy eigenvalues also becomes real, however, the potential (55) becomes *complex*, which is also obvious from (23). This is exactly the case of the $\mathcal{PT}$ invariant Scarf potential. $\mathcal{PT}$ invariant potentials are known to have real energy eigenvalues in certain circumstances, although they do not fulfil Hermiticity, but only the weakened condition of being invariant under simultaneous space ($\mathcal{P}$) and time ($\mathcal{T}$) reflection [15]: this corresponds to prescribing $V^*(-x) = V(x)$. The particular case of the $\mathcal{PT}$ invariant Scarf potential has been discussed in algebraic terms already [14, 26], however, then the $sl$(2,C) and $su(1,1)$ algebras were associated with it. Now it is obvious that these algebras can be embedded into larger algebras for the $\mathcal{PT}$ invariant version of this potential. The normalizable states of this potential will then belong to discrete irreducible representations of the relevant non-compact groups. In previous applications these were assigned to similar irreps of the $SO(2,1)$ or $SO(3)$ subgroups.

As an interesting recent development related to the $\mathcal{PT}$ invariant Scarf potential, it was shown that in certain situations this potential supports *no* (real-energy) bound states, rather all its energy eigenvalues are complex [28]. Using the present notation this corresponds to the relation $2|mm'| > m^2 + (m')^2$. This would require both $m$ and $m'$ to be imaginary, which indeed, corresponds to having complex-energy eigenstates for both sets of solutions. This problem has been discussed also in [29], where a systematic search was presented for $\mathcal{PT}$ invariant potentials having complex-energy solutions.

### *3.2. The Pöschl–Teller II potential*

As the second example, let us consider the $\mathcal{PT}$-symmetrized version of the Pöschl–Teller II potential (29), which, as we noted, is an alternative form of the generalized Pöschl–Teller potential (26). For this, let us introduce the imaginary coordinate shift $x \rightarrow x + \mathrm{i}\epsilon$, where $\epsilon$ is a real parameter [24]. This simple transformation can be interpreted in several ways. First, it is similar to other $\mathcal{PT}$-symmetric systems in that the problem is now defined on a contour of the complex plane [23]. However, when this contour is simply a straight line parallel to the $x$ axis, the imaginary coordinate shift can be interpreted as if the whole problem were defined on the real $x$ axis, and only the potential parameters had been defined to be complex [14].

In particular, the coupling coefficients of the even and odd potential components are real and imaginary, respectively, as they should be for $\mathcal{PT}$-symmetric problems.

The present subsection focuses on scattering aspects, while for the bound-state problem we refer to a recent work of Znojil [16].

The solution to the Schrödinger equation for the scattering states of potential (29) has been discussed in detail by various authors [11, 19, 30]. Since the potential is singular at the origin, the scattering solution is limited to the positive (or negative) half-line, thus yielding a transmission coefficient $T$ equal to zero, and a reflection coefficient $R$ of unit modulus.

The $\mathcal{PT}$-symmetric version is obtained by the replacement $x \to x + \mathrm{i}\epsilon$, which removes the singularity at the origin, provided $|\epsilon| < \pi/2$, so that both $R$ and $T$ are, in general, different from zero:

$$V(x) = -\left((m-m')^2 - \frac{1}{4}\right)\frac{1}{\cosh^2\alpha} + \left((m+m')^2 - \frac{1}{4}\right)\frac{1}{\sinh^2\alpha} \tag{62}$$

where $\alpha = x + \mathrm{i}\epsilon$ [32]. We treat the $\mathcal{PT}$-symmetric case in the same way as we did for the Scarf potential [14]. If one assumes that $m + m'$ is not integer, the two independent solutions can be written in terms of hypergeometric functions as

$$F_1(x) = \left(\frac{z-1}{2}\right)^{\frac{1}{2}\left(m+m'+\frac{1}{2}\right)}\left(\frac{z+1}{2}\right)^{\frac{1}{2}\left(m-m'+\frac{1}{2}\right)} \times F\left(\mathrm{i}\frac{k}{2} + m + \frac{1}{2}, -\mathrm{i}\frac{k}{2} + m + \frac{1}{2}; m + m' + 1; \frac{1-z}{2}\right) \tag{63}$$

and

$$F_2(x) = \left(\frac{z-1}{2}\right)^{\frac{1}{2}\left(-m-m'+\frac{1}{2}\right)}\left(\frac{z+1}{2}\right)^{\frac{1}{2}\left(m-m'+\frac{1}{2}\right)} \times F\left(\mathrm{i}\frac{k}{2} - m' + \frac{1}{2}, -\mathrm{i}\frac{k}{2} - m' + \frac{1}{2}; -m - m' + 1; \frac{1-z}{2}\right) \tag{64}$$

where $z = z(x) = \cosh(2(x + \mathrm{i}\epsilon))$. Note that (63) and (64) are obtained from each other by applying $m \leftrightarrow -m'$. (An alternative form of these solutions is presented in [19, 31] in terms of generalized Legendre functions.) The action of the ladder operators (30) and (31) on the algebraic versions of (63) and (64), $\Psi_1(m, m'; x) = \mathrm{e}^{\mathrm{i}(m\phi+m'\chi)}F_1(x)$ and $\Psi_2(m, m'; x) = \mathrm{e}^{\mathrm{i}(m\phi+m'\chi)}F_2(x)$ is similar to equations (58) to (61): $J_\pm$ and $K_\pm$ shift by one unit the $m$ and $m'$ labels, respectively.

Elementary quantum mechanics provides the reflection and transmission coefficients as functions of the asymptotic amplitudes of the progressive and regressive waves contained in $F_1$ and $F_2$ [30]:

$$\lim_{x\to\pm\infty} F_i(\alpha(x)) = a_{i\pm}\,\mathrm{e}^{\mathrm{i}kx} + b_{i\pm}\,\mathrm{e}^{-\mathrm{i}kx} \qquad (i = 1, 2) \tag{65}$$

$$T = \frac{a_{2+}b_{1+} - a_{1+}b_{2+}}{a_{2-}b_{1+} - a_{1-}b_{2+}} \tag{66}$$

$$R = \frac{b_{1+}b_{2-} - b_{1-}b_{2+}}{a_{2-}b_{1+} - a_{1-}b_{2+}}. \tag{67}$$

In order to compute the asymptotic coefficients $a_{i\pm}$ and $b_{i\pm}$, we exploit the following expansion of the hypergeometric function:

$$\lim_{|t|\to\infty} F(a, b, c; t) = \Gamma(c)\left(\frac{\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-t)^{-a} + \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-t)^{-b}\right). \tag{68}$$

The final result for the asymptotic coefficients of $F_1(x)$ is then

$$a_{1+} = 2^{-\mathrm{i}k}\,\mathrm{e}^{-k\epsilon}\frac{\Gamma(m+m'+1)\Gamma(\mathrm{i}k)}{\Gamma\left(\mathrm{i}\frac{k}{2}+\frac{1}{2}+m\right)\Gamma\left(\mathrm{i}\frac{k}{2}+\frac{1}{2}+m'\right)} \tag{69}$$

$$b_{1+} = 2^{\mathrm{i}k}\,\mathrm{e}^{k\epsilon}\frac{\Gamma(m+m'+1)\Gamma(-\mathrm{i}k)}{\Gamma\left(-\mathrm{i}\frac{k}{2}+\frac{1}{2}+m\right)\Gamma\left(-\mathrm{i}\frac{k}{2}+\frac{1}{2}+m'\right)} \tag{70}$$

$$a_{1-} = (-1)^{m+m'+\frac{1}{2}}2^{\mathrm{i}k}\,\mathrm{e}^{-k\epsilon}\frac{\Gamma(m+m'+1)\Gamma(-\mathrm{i}k)}{\Gamma\left(-\mathrm{i}\frac{k}{2}+\frac{1}{2}+m\right)\Gamma\left(-\mathrm{i}\frac{k}{2}+\frac{1}{2}+m'\right)} \tag{71}$$

$$b_{1-} = (-1)^{m+m'+\frac{1}{2}}2^{-\mathrm{i}k}\,\mathrm{e}^{k\epsilon}\frac{\Gamma(m+m'+1)\Gamma(\mathrm{i}k)}{\Gamma\left(\mathrm{i}\frac{k}{2}+\frac{1}{2}+m\right)\Gamma\left(\mathrm{i}\frac{k}{2}+\frac{1}{2}+m'\right)}. \tag{72}$$

The following relations hold:

$$b_{1-} = (-1)^{m+m'+\frac{1}{2}}\,\mathrm{e}^{2k\epsilon}a_{1+} \tag{73}$$

$$a_{1-} = (-1)^{m+m'+\frac{1}{2}}\,\mathrm{e}^{-2k\epsilon}b_{1+}. \tag{74}$$

According to the expression of $F_2(x)$ in (64), the $a_{2\pm}$ and $b_{2\pm}$ coefficients are immediately obtained from $a_{1\pm}$ and $b_{1\pm}$ by exchanging $m$ and $-m'$. Thus we have

$$b_{2-} = (-1)^{-m'-m+\frac{1}{2}}\,\mathrm{e}^{2k\epsilon}a_{2+} \tag{75}$$

$$a_{2-} = (-1)^{-m'-m+\frac{1}{2}}\,\mathrm{e}^{-2k\epsilon}b_{2+} \tag{76}$$

so $T$ and $R$ are obtained as

$$T = \mathrm{e}^{2k\epsilon}\frac{(-1)^{m'+m-\frac{1}{2}}}{1-(-1)^{2(m+m')}}\left(\frac{a_{2+}}{b_{2+}}-\frac{a_{1+}}{b_{1+}}\right) \tag{77}$$

$$R = \frac{\mathrm{e}^{4k\epsilon}}{1-(-1)^{2(m+m')}}\left[\frac{a_{2+}}{b_{2+}}-(-1)^{2(m+m')}\frac{a_{1+}}{b_{1+}}\right]. \tag{78}$$

Here

$$\frac{a_{2+}}{b_{2+}} = 2^{-2\mathrm{i}k}\,\mathrm{e}^{-2k\epsilon}\frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)}\frac{\Gamma\left(\frac{1}{2}-m'-\mathrm{i}\frac{k}{2}\right)\Gamma\left(\frac{1}{2}-m-\mathrm{i}\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}-m'+\mathrm{i}\frac{k}{2}\right)\Gamma\left(\frac{1}{2}-m+\mathrm{i}\frac{k}{2}\right)} \tag{79}$$

and

$$\frac{a_{1+}}{b_{1+}} = 2^{-2\mathrm{i}k}\,\mathrm{e}^{-2k\epsilon}\frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)}\frac{\Gamma\left(\frac{1}{2}+m-\mathrm{i}\frac{k}{2}\right)\Gamma\left(\frac{1}{2}+m'-\mathrm{i}\frac{k}{2}\right)}{\Gamma\left(\frac{1}{2}+m+\mathrm{i}\frac{k}{2}\right)\Gamma\left(\frac{1}{2}+m'+\mathrm{i}\frac{k}{2}\right)}. \tag{80}$$

With these the transmission coefficient (77) takes the form

$$T = \frac{2^{-2\mathrm{i}k}}{2\pi}\frac{\Gamma\left(m'+\frac{1}{2}-\mathrm{i}\frac{k}{2}\right)\Gamma\left(-m'+\frac{1}{2}-\mathrm{i}\frac{k}{2}\right)\Gamma\left(-m+\frac{1}{2}-\mathrm{i}\frac{k}{2}\right)\Gamma\left(m+\frac{1}{2}-\mathrm{i}\frac{k}{2}\right)}{\Gamma(1-\mathrm{i}k)\,\Gamma(-\mathrm{i}k)} \tag{81}$$

while the reflection coefficient (67) is expressed as

$$R = T\,\mathrm{i}\,\mathrm{e}^{2\epsilon k}\left[-\frac{\cos((m-m')\pi)}{\sinh(\pi k)}+\cos((m+m')\pi)(1-\coth(\pi k))\right]. \tag{82}$$

Note that while $T$ is independent of $\epsilon$, $R$ is not, similarly to our previous findings [14] concerning the Scarf (or Gendenshtein) potential discussed also in the previous subsection. Note also that for $m' = -m-\frac{1}{2}$ the Pöschl–Teller II potential (62) becomes the ordinary

Pöschl–Teller potential (more precisely, its $\mathcal{PT}$-symmetric version), thus $T$ in (81) also reduces to a simplified form

$$T_{m'=-m-\frac{1}{2}} = \frac{\Gamma(2m+1-\mathrm{i}k)\Gamma(-2m-\mathrm{i}k)}{\Gamma(1-\mathrm{i}k)\Gamma(-\mathrm{i}k)} \tag{83}$$

which, furthermore, has unit modulus whenever $m$ takes on integer or half-integer values. This demonstrates the well-known fact that for certain depths the ordinary Pöschl–Teller potential is reflectionless ($R = 0$), and in fact, this result also holds for the $\mathcal{PT}$-symmetric version of this potential as can be seen after substituting $m' = -m - \frac{1}{2}$ into (82) and then setting $m$ to integer or half-integer values. The same results follow also from the Pöschl–Teller limit of the Scarf potential, which is obtained for $\lambda = 0$ in the previous subsection and in [14].

It is to be stressed that, since $m$ and $m'$ are not simultaneously either integer or half-integer, then our wavefunctions cannot be classified in an ordinary irrep of the continuous principal series of $so(2,2)$ [12] with quantum number $\omega = -1 - \mathrm{i}k$, rather projective irreps [33] have to be considered then.

The bound-state energies are obtained from the poles of the transmission coefficient (81) and are naturally arranged as follows

$$2m' + 1 - \mathrm{i}k = -N^I \rightarrow E^I = k^2 = -(2m' + 2N^I + 1)^2 \tag{84}$$

$$-2m' + 1 - \mathrm{i}k = -N^{II} \rightarrow E^{II} = -(-2m' + 2N^{II} + 1)^2 \tag{85}$$

$$-2m + 1 - \mathrm{i}k = -N^{III} \rightarrow E^{III} = -(-2m + 2N^{III} + 1)^2 \tag{86}$$

$$2m + 1 - \mathrm{i}k = -N^{IV} \rightarrow E^{IV} = -(2m + 2N^{IV} + 1)^2. \tag{87}$$

In the above formulae the $N$ on the rhs are non-negative integers. Formula (84) holds for $m' < 0$ and implies a finite number of bound states, since $2N^I + 1 < -2m'$. Analogously, (85) holds for $m' > 0$, (86) for $m > 0$ and (87) for $m < 0$. In all cases the number of bound states is finite. Bound-state energies and wavefunctions agree with those of [16]: in particular, our hypergeometric functions reduce to Jacobi polynomials.

## 4. Conclusions

Starting with a general differential realization, we introduced an algebraic construction which includes the $so(2,2)$ and $so(4)$ algebras related to second-order differential equations. With an appropriate choice of the generators one can derive the hypergeometric differential equation or the Schrödinger equation with various potentials. In the latter case one obviously arrives at the Natanzon potentials, which have hypergeometric functions in their solution. The transformation between these differential equations is naturally performed by variable and similarity transformations, which leave the structure of the algebra invariant, except for changing compact algebras into non-compact ones (e.g. $so(3) \leftrightarrow so(2,1)$, etc). Our study extended the range of a similar work in which the differential realization of the $su(1,1) \sim so(2,1)$ algebra was considered.

We have shown that the two subalgebras are related to the two independent solutions of the second-order differential equations. We analysed a number of solvable shape-invariant potentials for which the $so(4)$ or $so(2,2)$ algebra appears as a potential algebra. These are the type A problems of the factorization method [7], for which the ladder operators are practically identical [8] with the SUSYQM operators which connect states belonging to the same energy but different potential strengths.

For Hermitian potentials only one of the solutions leads to bound (normalizable) states, and this explains why only the role of one of the two subalgebras has been emphasized in most of the previous applications. With the example of the hyperbolic Scarf potential we have shown that replacing Hermiticity with the requirement for $\mathcal{PT}$ invariance of the potential results in situations in which *both* solutions lead to normalizable states with real energy eigenvalues. Thus we have a problem where the carrier space of the $so(2,2)$ algebra is formed completely by normalizable states. We also demonstrated that the second set of solutions also appears naturally in the algebraic context for potentials which are singular at the origin in their Hermitian version, but can be regularized when they are required to be $\mathcal{PT}$-symmetric. We have shown that for the Pöschl–Teller II potential the transmission coefficients are independent of $\epsilon$, the parameter governing the complex coordinate shift, and that although this is not valid for the reflection coefficient in general, the ordinary Pöschl–Teller potential remains reflectionless for certain depths, even if this complex coordinate shift is applied to it.

We note that the $\mathcal{PT}$ symmetry approach of defining potentials on various trajectories of the complex $x$ plane actually unifies the PI-type potentials discussed in section 2.1, and allows their interpretation as manifestations of the *same* potential defined on various *lines* of the $x$ plane. Let us consider, for example, the generalized Pöschl–Teller potential (26) defined as the function of $z = u + \mathrm{i}v$. The $(u, v) = (x, \pi/2)$, $(x, 0)$, $(2x, 0)$, $(0, x)$ and $(0, 2x)$ choices lead to potentials containing the two terms as in (23), (26), (29), (32) and (35), respectively. Relaxing further the conditions regarding the complex coordinate, we can also easily introduce the $\mathcal{PT}$-symmetric versions of these potentials depending on the shifted coordinate $x + \mathrm{i}\epsilon$ [24].

In this paper we discussed only potential algebras, but the formalism followed here can be used to describe other structures, such as spectrum generating algebras too. Similarly to the $su(1,1) \sim so(2,1)$ case, this would correspond to taking $h_i(x) \neq$ const in (1) and (3) [8]. In this case the $j$ quantum number sets the potential strength via the $j(j+1)$ eigenvalue of the Casimir invariant, and the generators ladder between the states of the *same* potential. A particularly interesting aspect of this approach can be considering various versions of trigonometric Pöschl–Teller-like potentials as periodic problems. The singularities at the domain walls can cause a problem, however, as it was pointed out in [34], when the coupling strength of the $j(j+1)x^{-2}$-like terms (in fact, that of the $j(j+1)\sin^{-2}x$-like terms) is in the 'weakly attractive' domain, i.e. $-\frac{1}{4} < j(j+1) < 0$, then physically sound solutions can exist, which, furthermore, correspond to the unitary irreducible representations of $SU(1,1)$ called the supplementary series. Other types of the unitary irreducible representations of this group have already been associated with bound and scattering states of certain potentials in the potential algebra formalism [5], and this was the first case when the physical relevance of the supplementary series has been proposed [34]. More recently it was shown that these supplementary (or complementary) representations can be associated with the band spectrum of periodic potentials [35]. Considering that within the framework of $\mathcal{PT}$-symmetric quantum mechanics the singularities of periodic potentials of the Pöschl–Teller type can be cancelled in the standard way (see e.g. [36]), one expects that other types of irreducible representations can also be associated with the band-like spectra of these potentials.

We pointed out the importance of the second independent solution of the Schrödinger equation, which, in ordinary quantum mechanics, is irregular, except for the 'weakly attractive' and 'weakly repulsive' domains [37]. In $\mathcal{PT}$-symmetric quantum mechanics this second solution can be regularized, and this stresses the importance of the second set of ($so(2,1)$ or $so(3)$) algebra associated with these problems and the fact that these can be embedded into a larger algebra $so(2,2)$ or $so(4)$. This situation is clearly different from the one discussed in [11, 12], and it also has group theoretical implications, as discussed in section 3.1. There the second set of bound-state solutions corresponds to states that appear as resonances in the Hermitian version of the Scarf potential, and while in the Hermitian case these states are assigned to discrete non-unitary irreducible representations (see [25]), in the $\mathcal{PT}$-symmetric case they belong to discrete unitary irreps.

Similar considerations can be applied to the hypergeometric equation too: the action of the $J_\pm$ and $K_\pm$ operators on the second independent solution

$$\tilde{\psi}_{mm'}(z) = \mathrm{e}^{\mathrm{i}(m\phi+m'\chi)} z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z). \tag{88}$$

is the same as on (43). In fact, we find that the effect of these operators is the same as described in equations (49)–(52), only the roles of $J_i$ and $K_i$ (and also $m$ and $m'$) are exchanged. This interchangeability of the solutions and the algebras seems to be a general feature for all Natanzon potentials, which are related to the $so(2,2)$ algebra, because every single such potential can be derived from it in an algebraic fashion from the hypergeometric differential equation by variable and similarity transformations.

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